Bull. Cal. Math. Soc., **113,** (6) 493–510 (2021)

ENTIRE FUNCTIONS SHARING A LINEAR POLYNOMIAL WITH HIGHER ORDER DERIVATIVES OF LINEAR DIFFERENTIAL POLYNOMIAL

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(*Received 12 December 2021*)

Abstract. In the paper we study the uniqueness of entire functions sharing a linear polynomial with higher order derivatives of linear differential polynomials generated by them. The results of the paper improve and generalize the corresponding results of Lahiri-Kaish (J. Math. Anal. Appl. 406(2013), 66–74).

2010 AMS Subject Classifications : 30D35

Key words and phases : Entire function, Linear Differential Polynomial, Uniqueness.

1. Introduction, Definitions and Results. Let f be a nonconstant meromorphic function in the open complex plane $\mathbb{C}$ and a be a polynomial. We denote $E(a;f)$ the set of a-points of f, where each point is counted according its multiplicity. We denote by $\overline{E}(a;f)$ the reduced form of $E(a;f)$. For $A \subset \mathbb{C}$ we denote by $n_A(r,a;f)$ the number of zeros of $f-a$, counted with multiplicities, which lie in $A \cap \{z : |z| < r\}$. We define $N_A(r,a;f)$ as follows

$$N_A(r,a;f) = \int_0^r \frac{n_A(t,a;f) - n_A(0,a;f)}{t} dt + n_A(0,a;f) \log r.$$

Let f and g be two nonconstant meromorphic functions. We say that f and g share the polynomial a CM(counting multiplicities) if $E(a;f) = E(a;g)$. Also we say that f and g share a IM(ignoring multiplicities) if $\overline{E}(a;f) = \overline{E}(a;g)$. For standard definitions and results we refer the reader to (Hayman, 1964).

This paper was presented by the ICAHMMSMM–2019

In 1986 G. Jank, E. Mues and L. Volkman (1986) considered the case when an entire function shared a single value with its first two derivatives and proved the following result.

THEOREM A. (Jank, Mues and Volkman, 1986) *Let f be a nonconstant entire function and $a(\neq 0)$ be a finite number. If $\overline{E}(a;f) = \overline{E}(a;f^{(1)})$ and $\overline{E}(a;f) \subset \overline{E}(a;f^{(2)})$, then $f \equiv f^{(1)}$.*

In fact, in Theorem A, f and $f^{(1)}$ share the value a CM(counting multiplicities). Again considering $f = e^{wz} + w - 1$, where $w^{m-1} = 1, w \neq 1$ and $m(\geq 3)$ is an integer and $a = w$, we can verify that the second derivative in Theorem A can not be simply replaced by the m^{th} derivative for $m \geq 3$(see Zhong, 1995) .

In 1995 H. Zhong (1995) generalised Theorem A and proved the following theorem.

THEOREM B. (Zhong, 1995) *Let f be a non-constant entire function and $a(\neq 0)$ be a finite complex number. If f and $f^{(1)}$ share the value a CM and $\overline{E}(a;f) \subset \overline{E}(a;f^{(n)}) \cap \overline{E}(a;f^{(n+1)})$ for $n \geq 1$, then $f \equiv f^{(n)}$*

For $A \subset \mathbb{C} \cup \{\infty\}$, we denote by $N_A(r,a;f)(\overline{N}_A(r,a;f))$ the counting function (reduced counting function) of those a-points of f which belong to A.

In 2011 I. Lahiri and G. K. Ghosh (2011) improved Theorem B in the following manner.

THEOREM C. (Lahiri and Ghosh, 2011) *Let f be a nonconstant entire function and a, b be two nonzero finite constants. Suppose further that $A = \overline{E}(a;f) \setminus \overline{E}(a;f^{(1)})$ and $B = \overline{E}(a;f^{(1)}) \setminus \{\overline{E}(a;f^{(n)}) \cap \overline{E}(b;f^{(n+1)})\}$ for $n(\geq 1)$. If each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity and $N_A(r,a;f) + N_B(r,a;f^{(1)}) = S(r,f)$, then $f = \lambda e^{\frac{bz}{a}} + \frac{ab-a^2}{b}$ or $f = \lambda e^{\frac{bz}{a}} + a$, where $\lambda(\neq 0)$ is a constant.*

Throughout the paper we denote by L a nonconstant linear differential polynomial in f of the form

$$L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}, \tag{1.1}$$

where $a_1, a_2, \ldots, a_n(\neq 0)$ are constants.

In 1999 P. Li (1999) improved Theorem B by considering a linear differential polynomial instead of the derivative. The result of P. Li may be stated as follows:

THEOREM D. (Li, 1999) *Let f be nonconstant entire function and L be defined by* (1.1)*. If $\overline{E}(a;f)=\overline{E}(a;f^{(1)})$ and $\overline{E}(a;f)\subset\overline{E}(a;L)\cap\overline{E}(a;L^{(1)})$, then $f\equiv f^{(1)}\equiv L$.*

In the same paper P. Li (1999) also proved the following result.

THEOREM E. (Li, 1999) *Let f be a non-constant entire function and L be defined by* (1.1)*. If $\overline{E}(a;f)=\overline{E}(a;L)$, $\overline{E}(a;f)\subset\overline{E}(a;f^{(1)})\cap\overline{E}(a;L^{(1)})$ and $\sum\limits_{k=1}^{n}2^k a_k\neq 0$ or $\sum\limits_{k=1}^{n}a_k\neq -1$, then $f\equiv f^{(1)}\equiv L$.*

In 2011 I. Lahiri and G. K. Ghosh (2011) improved Theorem E by replacing the nature of sharing in the following manner.

THEOREM F. (Lahiri and Ghosh, 2011) *Let f be a non-constant entire function in $\mathbb{C}$, a be a finite nonzero complex number and L be defined by* (1.1)*. Further suppose that $E_{1)}(a;f)\subset E(a;f^{(1)})$ and $N_A(r,a;f)+N_B(r,a;L)=S(r,f)$, where $A=E(a;f)\setminus E(a;L)$ and $B=E(a;L)\setminus\{E(a;f^{(1)})\cap E(a;L^{(1)})\}$. Then one of the following cases holds:*

(i) *$f=a+\alpha e^z$ and $L=\alpha e^z$, where α is a nonzero constant*;

(ii) *$f=L=\alpha e^z$, where α is a nonzero constant;*

(iii) *$f=a+\frac{\alpha^2}{a}e^{2z}-\alpha e^z$ and $L=\alpha e^z$, where $\sum\limits_{k=1}^{n}2^k a_k=0$, $\sum\limits_{k=1}^{n}a_k=-1$ and α is a nonzero constant.*

In the same paper I. Lahiri and G. K. Ghosh also proved the following result.

THEOREM G. (Lahiri and Ghosh, 2011) *Let f be a nonconstant entire function in $\mathbb{C}$, a be a finite nonzero complex number and L be defined by* (1.1)*. Further let $N_A(r,a;f)+N_B(r,a;L)=S(r,f)$, where $A=E(a;f)\setminus E(a;L)$ and $B=E(a;L)\setminus\{E(a;f^{(1)})\cap E(a;L^{(1)})\}$. If $f\not\equiv L$ then one of the following holds:*

(i) *$f=a+\alpha e^z$ and $L=\alpha e^z$, where α is a nonzero constant;*

(ii) *$f=a+\frac{\alpha^2}{a}e^{2z}-\alpha e^z$ and $L=\alpha e^z$, where $\sum\limits_{k=1}^{n}2^k a_k=0$, $\sum\limits_{k=1}^{n}a_k=-1$ and α is a nonzero constant.*

In 2013 I. Lahiri and I. Kaish (2013) considered the case when f shares a nonzero finite value with $f^{(1)}$, $L^{(k)}$ and $L^{(k+1)}$ for some nonnegative integer k. They proved the following theorem.

THEOREM H. (Lahiri and Kaish, 2013) *Let f be a non-constant entire function, a be a finite nonzero complex number and $k(\geq 0)$ be an integer. Further suppose that L defined by* (1.1) *be such that $L^{(k+1)}$ is nonconstant and*

(i) $N_A(r,a;f) + N_B(r,a;L^{(k)}) + N_C(r,a;f^{(1)}) = S(r,f)$, *where* $A = \overline{E}(a;f) \setminus \overline{E}(a;L^{(k)})$, $B = \overline{E}(a;L^{(k)}) \setminus \{\overline{E}(a;f^{(1)}) \cap \overline{E}(a;L^{(k+1)})\}$ *and* $C = \overline{E}(a;f^{(1)}) \setminus \overline{E}(a;L^{(k+1)})$*;*

(ii) $\overline{E}_{1)}(a;f) \subset \overline{E}(a;f^{(1)}) \cap \overline{E}(a;L^{(k+1)})$ *and*

(iii) $\overline{E}_{(2}(a;f) \cap \overline{E}(0;L^{(k+1)}) = \emptyset$.

Then $L = \alpha e^z$ and $f = \alpha e^z$ or $f = a + \alpha e^z$, where $\alpha(\neq 0)$ is a constant.

Some interesting results on this topic have been obtained (see, e.g. (Kaish and I. Lahiri, 2018 and Wang, Lei and Chen, 2014)).

We now state the main result of the paper.

THEOREM 1.1 *Let f be a nonconstant entire function in $\mathbb{C}$, $a = \alpha z + \beta(\not\equiv f)$, where $\alpha(\neq 0)$ and β are constants , and L be defined by* (1.1)*.*

Further suppose that

(i) $N_A(r,a;f) + N_B(r,a;L^{(k)}) = S(r,f)$, *where* $A = \overline{E}(a;f) \setminus \overline{E}(a;L^{(k)})$ *and* $B = \overline{E}(a;L^{(k)}) \setminus \{\overline{E}(a;f^{(1)}) \cap \overline{E}(a;f^{(2)}) \cap \overline{E}(a;L^{(k+1)})\}$, *where $k(\geq 1)$ be an integer;*

(ii) $E_{1)}(a;f) \subset \overline{E}(a;f^{(1)})$ *and*

(iii) $\overline{N}_{(2}(r,a;f) = S(r,f)$.

Then $f = L = ce^z$ or $f = a + ce^z$, where $c(\neq 0)$ is a constant.

In the next theorem we see the possible form of an entire function if we drop the hypothesis $E_{1)}(a;f) \subset \overline{E}(a;f^{(1)})$. In fact the Case 2 of the proof of Theorem 1.1 suggests the following theorem.

THEOREM 1.2 *Let f be a nonconstant entire function in $\mathbb{C}$, $a = \alpha z + \beta (\not\equiv f)$, where $\alpha(\neq 0)$ and β are constants, and L be defined by* (1.1) . *Further let $N_A(r,a;f) + N_B(r,a;L^{(k)}) = S(r,f)$, where $k(\geq 1)$ be an integer and $\overline{N}_{(2}(r,a;f) = S(r,f)$, where $A = \overline{E}(a;f) \setminus \overline{E}(a;L^{(k)})$ and $B = \overline{E}(a;L^{(k)}) \setminus \{\overline{E}(a;f^{(1)}) \cap \overline{E}(a;f^{(2)}) \cap \overline{E}(a;L^{(k+1)})\}$. If $f \not\equiv L$ then $f = a + ce^z$, where $c(\neq 0)$ is a constant.*

The following example shows that the hypothesis (i) of Theorem 1.1. is essential.

EXAMPLE 1.1 *Let $f(z) = e^{(z)}$, $L = f^{(2)} + f^{(3)}$ and $a(z) = z$ then clearly $N_A(r,a;f) + N_B(r,a;L^{(k)}) \neq S(r,f)$ and $E_{1)}(a;f) \subset \overline{E}(a;f^{(1)})$. It is obvious that each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity, then $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$, we have that if z_0 is a common zero of f-a and $f^{(1)} - a$ with multiplicity $q(\geq 2)$, then z_0 is a zero of $a - a^{(1)}$ with multiplicity $q - 1$. So $N_{(2}(r,a;f) \leq 2N(r,0;a-a^{(1)}) + N_A(r,a;f) = S(r,f)$ that is hypothesis (iii) of Theorem 1.1. holds, but neither $f \equiv L$ nor $f = a + ce^z$.*

2. Lemmas. In this section we present some necessary lemmas.

LEMMA 2.1 {p.47 (Hayman, 1964)} *Let f be a nonconstant meromorphic function and a_1,a_2,a_3 be three distinct meromorphic functions satisfying $T(r,a_\mu) = S(r,f)$ for $\mu = 1,2,3$. Then*

$$T(r,f) \leq \overline{N}(r,0;f-a_1) + \overline{N}(r,0;f-a_2) + \overline{N}(r,0;f-a_3) + S(r,f).$$

LEMMA 2.2 {p.57 (Hayman, 1964)} *Suppose that g is a nonconstant meromorphic function and $\Psi = \sum\limits_{\mu=0}^{l} a_\mu g^{(\mu)}$ where $a_\mu^{,}s$ are meromorphic functions satisfying $T(r,a_\mu) = S(r,g)$ for $\mu = 0,1,2,\cdots,l$. If Ψ is nonconstant, then*

$$T(r,g) \leq \overline{N}(r,\infty;g) + N(r,0;g) + \overline{N}(r,1;\Psi) + S(r,g).$$

LEMMA 2.3 *Let f be a transcendental meromorphic function and $a = \alpha z + \beta$, where $\alpha(\neq 0)$ and β are constants. Then*

$$T(r,f) \leq \overline{N}(r,\infty;f) + N(r,a;f) + \overline{N}(r,a;L^{(k)}) + S(r,f).$$

Proof: The lemma follows from Lemma 2.2 for $g = f - a, a_0 = 0$ and $\Psi = \frac{L^{(k)}}{a}$. This proves the lemma.

LEMMA 2.4 {p.68 (Hayman, 1964)} *Let f be meromorphic and transcendental function in $\mathbb{C}$ and $f^n P = Q$, where P, Q are differential polynomials in f and the degree of Q is at most n. Then $m(r, P) = S(r, f)$.*

LEMMA 2.5 *Let f be a transcendental entire function and L defined by* (1.1) *be such that $L^{(k+1)}$ is nonconstant. Let $a = \alpha z + \beta$, where $\alpha(\neq 0)$ and β are constants and $m(r, a; f) = S(r, f)$. Further suppose that*

(i) *$N_A(r, a; f) + N_B(r, a; L^{(k)}) = S(r, f)$, where $A = \overline{E}(a; f) \setminus \overline{E}(a; L^{(k)})$ and $B = \overline{E}(a; L^{(k)}) \setminus \{\overline{E}(a; f^{(1)}) \cap \overline{E}(a; f^{(2)}) \cap \overline{E}(a; L^{(k+1)})\}$, where $k(\geq 1)$ be an integer;*

(ii) *$E_{1)}(a; f) \subset \overline{E}(a; f^{(1)})$. Then $f \equiv L \equiv ce^z$, where c is a nonzero constant.*

Proof: Let

$$\gamma = \frac{f^{(1)} - a}{f - a}. \tag{2.1}$$

From the hypotheses we see that γ has no simple pole and

$$\begin{aligned} N(r, \gamma) &\leq N_A(r, a; f) + N_B(r, a; L^{(k)}) + S(r, f) \\ &= S(r, f) \end{aligned}$$

and since $m(r, a; f) = S(r, f)$ we get

$$\begin{aligned} m(r, \gamma) &= m(r, \frac{f^{(1)} - a}{f - a}) \\ &= m(r, \frac{f^{(1)} - a^{(1)}}{f - a} + \frac{a^{(1)} - a}{f - a}) + S(r, f) \\ &\leq m(r, \frac{a^{(1)} - a}{f - a}) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Hence $T(r,\gamma)=S(r,f)$. From (2.1) we get

$$f^{(1)}=\gamma_1 f+\mu_1, \tag{2.2}$$

where $\gamma_1=\gamma$ and $\mu_1=a(1-\gamma)$.

We repeat the above argument $(j-1)$-times by differentiating (2.2) we get

$$f^{(j)}=\gamma_j f+\mu_j (j=1,2,\ldots), \tag{2.3}$$

where γ_j and μ_j are meromorphic functions satisfying $\gamma_j=\gamma_{j-1}^{(1)}+\gamma_1\gamma_{j-1}$ and $\mu_j=\mu_{j-1}^{(1)}+\mu_1\gamma_{j-1}$ for $j=1,2,\ldots$. Also we note that $T(r,\gamma_j)+T(r,\mu_j)=S(r,f)$ for $j=1,2,\ldots$.

Now

$$L^{(k)}=\sum_{j=1}^{n}a_j f^{(j+k)}=\left(\sum_{j=1}^{n}a_j\gamma_{j+k}\right)f+\sum_{j=1}^{n}a_j\mu_{j+k}=\xi f+\eta,\ \text{say}. \tag{2.4}$$

Clearly $T(r,\xi)+T(r,\eta)=S(r,f)$. Differentiating (2.4) we get

$$L^{(k+1)}=\xi f^{(1)}+\xi^{(1)}f+\eta^{(1)}. \tag{2.5}$$

Let z_0 be a simple zero of $f-a$ such that $z_0\notin A\cup B\cup C$ where $C=\{z : a(z)-a^{(1)}(z)=0\}$. Then from (2.4) and (2.5) we get $a(z_0)\xi(z_0)+\eta(z_0)=a(z_0)$ and $a(z_0)\xi(z_0)+a(z_0)\xi^{(1)}(z_0)+\eta^{(1)}(z_0)=a(z_0)$. First suppose that $a\xi+\eta\not\equiv a$. Since every multiple zero of $f-a$ must belong to $A\cup B\cup C$ then we get

$$\begin{aligned}N(r,a;f)&\le N_A(r,a;f)+N_B(r,a;L^{(k)})+N(r,a;a\xi+\eta)\\&=S(r,f),\end{aligned}$$

which is impossible because $m(r,a;f)=S(r,f)$. Hence

$$a\xi+\eta\equiv a. \tag{2.6}$$

Similarly

$$a\xi+a\xi^{(1)}+\eta^{(1)}\equiv a. \tag{2.7}$$

Differentiating (2.6) and then subtract (2.7) we get $a-a^{(1)}=\xi(a-a^{(1)})$. Since $a\not\equiv a^{(1)}$ we get $\xi\equiv 1$ and $\eta\equiv 0$. Then from (2.4) we get $f\equiv L^{(k)}$.

By actual calculation we see that $\gamma_2 = \gamma^2 + \gamma^{(1)}$ and $\gamma_3 = \gamma^3 + 3\gamma\gamma^{(1)} + \gamma^{(2)}$. In general, we now verify that

$$\gamma_{j+k} = \gamma^{j+k} + P_{j+k-1}[\gamma], \tag{2.8}$$

where $P_{j+k-1}[\gamma]$ is a differential polynomial in γ with constant coefficients having degree at most $j+k-1$ and weight at most $j+k$. Also we note that each term of $P_{j+k-1}[\gamma]$ contains some derivative of γ.

Let (2.8) be true. Then

$$\begin{aligned}\gamma_{j+k+1} &= \gamma_{j+k}^{(1)} + \gamma_1\gamma_{j+k}\\ &= (\gamma^{j+k} + P_{j+k-1}[\gamma])^{(1)} + \gamma(\gamma^{j+k} + P_{j+k-1}[\gamma])\\ &= \gamma^{j+k+1} + (j+k)\gamma^{j+k-1}\gamma^{(1)} + (P_{j+k-1}[\gamma])^{(1)} + \gamma P_{j+k-1}[\gamma]\\ &= \gamma^{j+k+1} + P_{j+k}[\gamma],\end{aligned}$$

noting that differentiation does not increase the degree of a differential polynomial but increase its weight by 1. So (2.8) is verified by mathematical induction.

Since $\sum\limits_{j=1}^{n} a_j\gamma_{j+k} = \xi = 1$, we get from (2.8)

$$\sum_{j=1}^{n} a_j\gamma^{j+k} + \sum_{j=1}^{n} a_j P_{j+k-1}[\gamma] \equiv 1. \tag{2.9}$$

If z_0 is a pole of γ with multiplicity $p(\geq 2)$, then z_0 is a pole of $\sum\limits_{j=1}^{n} a_j\gamma^{j+k}$ with multiplicity $(n+k)p$ and it is a pole of $\sum\limits_{j=1}^{n} a_k P_{j+k-1}[\gamma]$ with multiplicity not exceeding $(n+k-1)p+1$. Since $(n+k)p > (n+k-1)p+1$, it follows that z_0 is a pole of the left hand side of (2.9) with multiplicity $(n+k)p$, which is impossible. So γ is an entire function. If γ is transcendental, from (2.9) we get by Lemma 2.4 that $m(r,\gamma) = S(r,\gamma)$ and if γ is a polynomial then following the proof of Lemma 2.4 we get $m(r,\gamma) = O(1)$. Therefore γ is a constant. Hence from (2.9) we obtain $\gamma_{j+k} = \gamma^{j+k}$ for $j = 1, 2, \ldots$.

Since $\xi \equiv 1$, we see that $\sum\limits_{j=1}^{n} a_j\gamma^{j+k} \equiv 1$. Also from (2.2) we obtain $f^{(1)} = \gamma f + a(1-\gamma)$ then $f^{(2)} = \gamma f^{(1)} + \alpha(1-\gamma)$ and $f^{(3)} = \gamma f^{(2)}$ and so $f^{(2)} = ce^{\gamma z}$, where $c(\neq 0)$ is a constant. Then $f^{(1)} = \frac{ce^{\gamma z}}{\gamma} + d$. Since $L^{(k)} \equiv f$ so $L^{(k+1)} \equiv f^{(1)}$

implies $d + \frac{ce^{\gamma z}}{\gamma} = L^{(k+1)} = a_1 f^{(k+2)} + a_2 f^{(k+3)} + \cdots + a_n f^{(k+n+1)} = ce^{\gamma z}(a_1\gamma^k + a_2\gamma^{k+1} + \cdots + a_n\gamma^{k+n-1})$ then $d = 0$ and $\sum_{j=1}^{n} a_j\gamma^{j+k} = 1$. So $f^{(1)} = \frac{ce^{\gamma z}}{\gamma}$ and $f = \frac{ce^{\gamma z}}{\gamma^2} + d_1$. Since $m(r, a; f) = S(r, f)$ then obviously $N(r, a; f) \neq S(r, f)$. By hypothesis $N_A(r, a; f) + N_B(r, a; L^{(k)}) = S(r, f)$ so $E(a; f) \cap E(a; f^{(1)}) \neq \emptyset$. Hence from $f^{(1)} = \frac{ce^{\gamma z}}{\gamma}$ and $f = \frac{ce^{\gamma z}}{\gamma^2} + d_1$ we get $d_1 = 0$ and $\gamma = 1$. Hence $L \equiv f \equiv ce^z$.

3. Proof of the theorem

Proof of Theorem 1.1 First we claim that f can not be a polynomial. If f is a polynomial, then $T(r, f) = O(\log r)$. Since f is a polynomial so $f - a$ and $L^{(k)} - a$ have only finite number of zeros. If $A \neq \emptyset$ then A contains finite number of zeros of $f - a$. Then $N_A(r, a; f) = O(\log r)$, similarly if $B \neq \emptyset$ then $N_B(r, a; L^{(k)}) = O(\log r)$ and so $N_A(r, a; f) + N_B(r, a; L^{(k)}) = O(\log r)$. But by the hypothesis $N_A(r, a; f) + N_B(r, a; L^{(k)}) = S(r, f)$. Therefore $T(r, f) = O(\log r) = S(r, f)$, a contradiction. Hence $A = B = \emptyset$. Therefore $\overline{E}(a; f) \subset \overline{E}(a; L^{(k)}) \subset \overline{E}(a; f^{(1)}) \cap \overline{E}(a; f^{(2)}) \cap \overline{E}(a; L^{(k+1)})$.

First we suppose that degree of f be 1 and we consider $f = A_1 z + B_1$, where $A_1 (\neq 0), B_1$ are constants. Then $f^{(1)} = A_1, f^{(2)} = 0, L^{(k)} = a_1 f^{(k+1)} + a_2 f^{(k+2)} + \cdots + a_n f^{(k+n)} = 0 = L^{(k+1)}$. Now $f - a = A_1 z + B_1 - \alpha z - \beta = 0$, implies $z = \frac{\beta - B_1}{A_1 - \alpha}$ is the only zero of $f - a$, $\frac{A_1 - \beta}{\alpha}$ is the only zero of $f^{(1)} - a$ and $-\frac{\beta}{\alpha}$ is the only zero of $L^{(k)} - a$ and also since $\overline{E}(a; L^{(k)}) \subset \overline{E}(a; f^{(1)})$ so, $\frac{A_1 - \beta}{\alpha} = -\frac{\beta}{\alpha}$ implies $A_1 = 0$, which is a contradiction.

We denote by $N_{(2}(r, a; f \mid L^{(k)} = a)$ the counting function (counted with multiplicities) of those multiple a-points of f which are a-points of $L^{(k)}$. We first note that

$$\begin{aligned} N_{(2}(r, a; f) &\leq N_A(r, a; f) + N_{(2}(r, a; f \mid L^{(k)} = a) \\ &\leq (n + k)\overline{N}_{(2}(r, a; f) + S(r, f) \\ &= S(r, f). \end{aligned}$$

Now let f be a polynomial of degree greater than 1. Since $N_{(2}(r, a; f) = S(r, f)$, we see that $f - a$ has no multiple zero and so all the zeros of $f - a$ are distinct. Since $\overline{E}(a; f) \subset \overline{E}(a; f^{(1)})$ and $deg(f - a) = deg(f^{(1)} - a) + 1$, we arrive at a contradiction.

Therefore f is a transcendental entire function. Now we divide our argument into the following cases.

CASE 1. $f \equiv L^{(k)}$. Then $f^{(1)} \equiv L^{(k+1)}$. Now

$$\begin{aligned} m(r,a;f) &= m\left(r, \frac{1}{f-a}\right) \\ &= m(r, \frac{f^{(1)}-a^{(1)}}{f-a} \cdot \frac{1}{f^{(1)}-a^{(1)}}) \\ &\le m\left(r, \frac{1}{f^{(1)}-a^{(1)}}\right) + S(r,f) \\ &\le m\left(r, \frac{a^{(1)}}{f^{(1)}-a^{(1)}} + 1\right) + S(r,f) \\ &= m\left(r, \frac{L^{(k+1)}}{f^{(1)}-a^{(1)}}\right) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{3.1}$$

So by Lemma 2.5 we get $L \equiv f \equiv ce^z$ where $c(\neq 0)$ is a constant.

CASE 2. $f \not\equiv L^{(k)}$. Then we consider following subcases:

SUBCASE 2.1. Suppose that $L^{(k+1)} \not\equiv f^{(1)}$. Here we have to consider following subcases.

SUBCASE 2.1.1. Suppose $L^{(k)} \equiv L^{(k+1)}$ and $L^{(k)} \not\equiv f^{(1)}$. Then we have two possibilities either $L^{(k)} \equiv L^{(k+1)}$ and $L^{(k+1)} \equiv f^{(2)}$ or $L^{(k)} \equiv L^{(k+1)}$ and $L^{(k+1)} \not\equiv f^{(2)}$.

If we consider the possibility $L^{(k)} \equiv L^{(k+1)}$ and $L^{(k+1)} \equiv f^{(2)}$. Then $L^{(k)} \equiv L^{(k+1)}$ implies $L^{(k)} = ce^z$(c is a non zero constant) and so $L^{(k+1)} = f^{(2)} = ce^z$ then $f^{(1)} = ce^z + \lambda$, and $f = ce^z + \lambda z + \delta$. Since $L^{(k)} \neq f^{(1)}$ obviously $\lambda \neq 0$.

If we consider $\lambda z + \delta \neq a$. Then by Lemma 2.1 we get

$$\begin{aligned} T(r, ce^z) &\le \overline{N}(r, 0; ce^z) + \overline{N}(r, \infty; ce^z) + \overline{N}(r, a - \lambda z - \delta; ce^z) + S(r, ce^z) \\ &= \overline{N}(r, a; f) + S(r, ce^z). \end{aligned} \tag{3.2}$$

Since $f = L^{(k+1)} + \lambda z + \delta$, we see that if z_1 is a zero of $f - a$ such that $z_1 \notin A \cup B$

then $\lambda z+\delta=0$. Therefore

$$\begin{aligned}\overline{N}(r,a;f) &\le N_A(r,a;f)+N_B(r,a;L^{(k)})+N(r,0;\lambda z+\delta)\\ &= S(r,f).\end{aligned}$$

which contradicts (3.2).

Next we consider $\lambda z+\delta\equiv a$, then $f=ce^z+a$ and so $f^{(1)}=ce^z+\alpha$ and $f^{(2)}=ce^z$.

Hence $L^{(k)}=(a_1+a_2+\cdots+a_n)ce^z=f^{(2)}=ce^z$ implies $\sum\limits_{j=1}^{n}a_j=1$. Hence we get $L^{(k)}=L^{(k+1)}=ce^z$ and $f=a+ce^z$ where $c(\ne 0)$ is a constant and $\sum\limits_{j=1}^{n}a_j=1$.

Next we consider the possibility $L^{(k)}\equiv L^{(k+1)}$ and $L^{(k+1)}\not\equiv f^{(2)}$. Hence $L^{(k)}\not\equiv f^{(2)}$. Then by the hypothesis we get

$$\begin{aligned}\overline{N}(r,a;L^{(k)}) &\le N_B(r,a;L^{(k)})+N\left(r,1;\frac{L^{(k)}}{f^{(2)}}\right)\\ &\le T\left(r,\frac{L^{(k)}}{f^{(2)}}\right)+S(r,f)\\ &= N\left(r,\frac{L^{(k)}}{f^{(2)}}\right)+S(r,f)\\ &\le N(r,0;f^{(2)})+S(r,f).\end{aligned}\tag{3.3}$$

Again

$$\begin{aligned}m(r,a;f) &= m\left(r,\frac{f^{(2)}}{f-a}\cdot\frac{1}{f(2)}\right)\\ &\le m(r,0;f^{(2)})+S(r,f)\\ &= T(r,f^{(2)})-N(r,0;f^{(2)})+S(r,f)\\ &= m(r,f^{(2)})-N(r,0;f^{(2)})+S(r,f)\\ &\le m(r,f)-N(r,0;f^{(2)})+S(r,f)\\ &= T(r,f)-N(r,0;f^{(2)})+S(r,f)\end{aligned}$$

and so

$$N(r,0;f^{(2)}) \le N(r,a;f) + S(r,f). \tag{3.4}$$

Hence from (3.3) and (3.4) we get

$$\overline{N}(r,a;L^{(k)}) \le N(r,a;f) + S(r,f), \tag{3.5}$$

which implies by Lemma 2.3 that

$$T(r,f) \le 2N(r,a;f) + S(r,f). \tag{3.6}$$

We put $\quad \Phi = \dfrac{f^{(2)} - L^{(k)}}{f-a} \quad$ and $\quad \Psi = \dfrac{(a-a^{(1)})f^{(2)} - a(f^{(1)} - a^{(1)})}{f-a}.$

Then

$$\begin{aligned} N(r,\Phi) &\le N_A(r,a;f) + N_B(r,a;L^{(k)}) + N_{(2}(r,a;f) + S(r,f) \\ &= S(r,f), \end{aligned}$$

also $N(r,\Psi) = S(r,f)$, and $m(r,\Phi) = S(r,f)$, $m(r,\Psi) = S(r,f)$. Therefore $T(r,\Phi) = S(r,f)$ and $T(r,\Psi) = S(r,f)$.

Since $L^{(k)} \not\equiv f^{(2)}$ so $\Phi \not\equiv 0$.

Let z_2 be a simple zero of $f-a$ such that $z_2 \notin A \cup B \cup C$ where $C = \{z : a(z) - a^{(1)}(z) = 0\}$.

Then by Taylor's expansion in some neighbourhood of z_2 we get

$$\begin{aligned} f - a &= (f-a)(z_2) + (f-a)^{(1)}(z_2)(z-z_2) + (f-a)^{(2)}(z_2)\frac{(z-z_2)^2}{2} \\ &\quad + (f-a)^{(3)}(z_2)\frac{(z-z_2)^3}{6} + \dots \\ &= (a(z_2) - a^{(1)}(z_2))(z-z_2) + a(z_2)\frac{(z-z_2)^2}{2} + f^{(3)}(z_2)\frac{(z-z_2)^3}{6} + \dots \end{aligned}$$

Now differentiating we obtain

$$f^{(1)} - \alpha = a(z_2) - a^{(1)}(z_2) + a(z_2)(z-z_2) + f^{(3)}(z_2)\frac{(z-z_2)^2}{2} + \dots$$

and

$$f^{(2)} = a(z_2) + f^{(3)}(z_2)(z - z_2) + \dots$$

Also,

$$\begin{aligned} L^{(k)} &= L^{(k)}(z_2) + L^{(k+1)}(z_2)(z - z_2) + L^{(k+2)}(z_2)\frac{(z-z_2)^2}{2} + \dots \\ &= a(z_2) + a(z_2)(z - z_2) + L^{(k+2)}(z_2)\frac{(z-z_2)^2}{2} + \dots \end{aligned}$$

Therefore in some neighbourhood of z_2 we get

$$\begin{aligned} \Phi(z) &= \frac{a(z_2) + f^{(3)}(z_2)(z-z_2) - a(z_2) - a(z_2)(z-z_2) + O(z-z_2)^2}{(a(z_2)-\alpha)(z-z_2) + O(z-z_2)^2} \\ &= \frac{(f^{(3)}(z_2) - a(z_2))(z-z_2) + O(z-z_2)^2}{(a(z_2)-\alpha)(z-z_2) + O(z-z_2)^2} \\ &= \frac{f^{(3)}(z_2) - a(z_2) + O(z-z_2)}{a(z_2) - \alpha + O(z-z_2)} \end{aligned}$$

Noting that $a(z_2) - \alpha \neq 0$, then

$$\Phi(z_2) = \frac{f^{(3)}(z_2) - a(z_2)}{a(z_2) - \alpha}. \tag{3.7}$$

Also in some neighbourhood of z_2 we get

$$\begin{aligned} \Psi(z) &= \frac{\begin{array}{r}\{a(z) - a^{(1)}(z)\}\{a(z_2) + f^{(3)}(z_2)(z-z_2)\} - a(z)\{a(z_2) \\ -a^{(1)}(z_2) + a(z_2)(z-z_2)\} + O(z-z_2)^2\end{array}}{(a(z_2)-\alpha)(z-z_2) + O(z-z_2)^2} \\ &= \frac{\alpha^2(z-z_2) + \{(a(z)-\alpha)f^{(3)}(z_2) - a(z)a(z_2)\}(z-z_2) + O(z-z_2)^2}{(a(z_2)-\alpha)(z-z_2) + O(z-z_2)^2} \\ &= \frac{\alpha^2 + (a(z)-\alpha)f^{(3)}(z_2) - a(z)a(z_2) + O(z-z_2)}{a(z_2) - \alpha + O(z-z_2)} \end{aligned}$$

Hence

$$\begin{aligned} \Psi(z_2) &= \frac{(f^{(3)}(z_2) - a(z_2) - \alpha)(a(z_2) - \alpha)}{a(z_2) - \alpha} \\ &= f^{(3)}(z_2) - a(z_2) - \alpha \end{aligned} \tag{3.8}$$

From (3.7) and (3.8) we get

$$(a(z_2) - \alpha)\Phi(z_2) = \Psi(z_2) + a(z_2) + \alpha - a(z_2)$$

implies

$$(a(z_2) - \alpha)\Phi(z_2) - \Psi(z_2) - \alpha = 0.$$

If

$$(a - \alpha)\Phi - \Psi - \alpha \not\equiv 0,$$

then we get

$$\begin{aligned} N(r, a; f) &\leq N_A(r, a; f) + N_B(r, a; L^{(k)}) + N_{(2}(r, a; f) \\ &\quad + N(r, 0; (a - \alpha)\Phi - \Psi - \alpha) \\ &= S(r, f), \end{aligned}$$

which contradicts (3.6).

Therefore

$$(a - \alpha)\Phi - \Psi - \alpha \equiv 0. \tag{3.9}$$

First we suppose that $\Psi \equiv 0$. Then from (3.9) and the definitions of Φ and Ψ we get $(a - \alpha)\frac{f^{(2)} - L^{(k)}}{f - a} = \alpha$ and $(a - \alpha)f^{(2)} - a(f^{(1)} - \alpha) = 0$ implies

$$(a - \alpha)f^{(2)} - (a - \alpha)L^{(k)} = \alpha(f - a) \tag{3.10}$$

and

$$(a - \alpha)f^{(2)} = a(f^{(1)} - \alpha). \tag{3.11}$$

From (3.10) and (3.11) we get

$$a(f^{(1)} - \alpha) - (a - \alpha)L^{(k)} = \alpha(f - a). \tag{3.12}$$

Differentiating (3.12) we get

$$af^{(2)} + \alpha(f^{(1)} - \alpha) - \alpha L^{(k)} - (a - \alpha)L^{(k+1)} = \alpha(f^{(1)} - \alpha). \tag{3.13}$$

Since $L^{(k)} \equiv L^{(k+1)}$ then from (3.13) we get $af^{(2)} = aL^{(k)}$ implies $a(f^{(2)} - L^{(k)}) = 0$, since $a \neq 0$ so $f^{(2)} - L^{(k)} \equiv 0$ and so $\Phi \equiv 0$, which is a contradiction.

Next we suppose that $\Psi \not\equiv 0$. Then from (3.9) and the definitions of Φ and Ψ we get

$$(a-\alpha)\frac{f^{(2)}-L^{(k)}}{f-a}-\frac{(a-\alpha)f^{(2)}-a(f^{(1)}-\alpha)}{f-a}=\alpha$$

this implies

$$-(a-\alpha)L^{(k)}+a(f^{(1)}-\alpha)=\alpha(f-a). \tag{3.14}$$

Differentiating both sides of (3.14) and put $L^{(k)} \equiv L^{(k+1)}$ we get $a(L^{(k)}-f^{(2)})=0$, since $a \neq 0$ so $f^{(2)}-L^{(k)} \equiv 0$ and so $\Phi \equiv 0$, which is a contradiction.

SUBCASE 2.1.2 Let $L^{(k)} \not\equiv L^{(k+1)}$ and $L^{(k)} \equiv f^{(1)}$.

We put $$\tau=\frac{(a-a^{(1)})L^{(k)}-a(f^{(1)}-a^{(1)})}{f-a}.$$

Then

$$\begin{aligned} N(r,\tau) &\le N_A(r,a;f)+N_B(r,a;L^{(k)})+N_{(2}(r,a;f)+S(r,f)\\ &= S(r,f), \end{aligned}$$

also $m(r,\tau)=S(r,f)$. Therefore $T(r,\tau)=S(r,f)$.

Let z_3 be a simple zero of $f-a$ such that $z_3 \notin A\cup B\cup C$ where $C=\{z : a(z)-a^{(1)}(z)=0\}$.

Then by Taylor's expansion in some neighbourhood of z_3 we get

$$\begin{aligned} f-a &= (f-a)(z_3)+(f-a)^{(1)}(z_3)(z-z_3)\\ &\quad +(f-a)^{(2)}(z_3)\frac{(z-z_3)^2}{2}+O(z-z_3)^3\\ &= (a(z_3)-\alpha)(z-z_3)+a(z_3)\frac{(z-z_3)^2}{2}+O(z-z_3)^3 \end{aligned}$$

Now differentiating we obtain

$$f^{(1)}-\alpha=(a(z_3)-\alpha)+a(z_3)(z-z_3)+(z-z_3)^2$$

and

$$\begin{aligned} L^{(k)} &= L^{(k)}(z_3)+L^{(k+1)}(z_3)(z-z_3)+O(z-z_3)^2\\ &= a(z_3)+a(z_3)(z-z_3)+O(z-z_3)^2 \end{aligned}$$

Therefore in some neighbourhood of z_3 we get

$$\begin{aligned}\tau(z) &= \frac{\begin{array}{r}\{a(z)-a^{(1)}(z)\}\{a(z_3)+a(z_3)(z-z_3)\}-a(z)\{a(z_3)\\ -\alpha+a(z_3)(z-z_3)\}+O(z-z_3)^2\end{array}}{(a(z_3)-\alpha)(z-z_3)+O(z-z_3)^2}\\ &= \frac{\alpha^2(z-z_3)-\alpha a(z_3)(z-z_3)+O(z-z_3)^2}{(z-z_3)(a(z_3)-\alpha+O(z-z_3))}\\ &= \frac{-\alpha(a(z_3)-\alpha)+O(z-z_3)}{a(z_3)-\alpha+O(z-z_3)}\\ &= -\alpha+O(z-z_3)\end{aligned}$$

Let $P=\tau+\alpha$. Then in some neighbourhood of z_3 we get $P(z)=O(z-z_3)$.

First we suppose that $P(z)\not\equiv 0$. Since every multiple zero of $f-a$ must belongs to $A\cup B\cup C$, then we get

$$\begin{aligned}N(r,a;f) &\le N_A(r,a;f)+N_B(r,a;L^{(k)})+N(r,0;P)\\ &= S(r,f).\end{aligned}$$

Then from (3.17) we get $T(r,f)=S(r,f)$, a contradiction. Hence $P\equiv 0$ and so

$$(a-\alpha)L^{(k)}-a(f^{(1)}-\alpha)+\alpha(f-a)=0.$$

Since $L^{(k)}\equiv f^{(1)}$ then we get

$$(a-\alpha)f^{(1)}-a(f^{(1)}-\alpha)+\alpha(f-a)=0$$

which implies $\alpha(f-f^{(1)})=0$, since $\alpha\neq 0$ then $f\equiv f^{(1)}$. So $f=ce^z$ where $c(\neq 0)$ is a constant. Then

$$\begin{aligned}L^{(k)} &= a_1f^{(k+1)}+a_2f^{(k+2)}+\cdots+a_nf^{(k+n)}\\ &= (a_1+a_2+\cdots+a_n)ce^z\\ \text{and }\ L^{(k+1)} &= a_1f^{(k+2)}+a_2f^{(k+3)}+\cdots+a_nf^{(k+n+1)}\\ &= (a_1+a_2+\cdots+a_n)ce^z.\end{aligned}$$

So $L^{(k)}\equiv L^{(k+1)}$ which is a contradiction.

SUBCASE 2.2. Let $L^{(k)}\not\equiv L^{(k+1)}$ and $L^{(k+1)}\equiv f^{(1)}$. Since $L^{(k)}\not\equiv L^{(k+1)}$. Then by hypothesis we get

$$\begin{aligned}\overline{N}(r,a;L^{(k)}) &\le N_B(r,a;L^{(k)}) + N\left(r,1;\frac{L^{(k+1)}}{L^{(k)}}\right)\\ &\le T\left(r,\frac{L^{(k+1)}}{L^{(k)}}\right) + S(r,f)\\ &= N\left(r,\frac{L^{(k+1)}}{L}\right) + S(r,f)\\ &\le N(r,0;L^{(k)}) + S(r,f).\end{aligned} \tag{3.15}$$

Again
$$\begin{aligned}m(r,a;f) &= m\left(r,\frac{L^{(k)}}{f-a}.\frac{1}{L^{(k)}}\right) \le m(r,0;L^{(k)}) + S(r,f)\\ &= T(r,L^{(k)}) - N(r,0;L^{(k)}) + S(r,f)\\ &= m(r,L^{(k)}) - N(r,0;L^{(k)}) + S(r,f)\\ &\le m(r,f) - N(r,0;L^{(k)}) + S(r,f)\\ &= T(r,f) - N(r,0;L^{(k)}) + S(r,f)\end{aligned}$$

and so
$$N(r,0;L^{(k)}) \le N(r,a;f) + S(r,f). \tag{3.16}$$

Hence from (3.15) and (3.16) we get
$$\overline{N}(r,a;L^{(k)}) \le N(r,a;f) + S(r,f),$$

which implies by Lemma 2.3 that
$$T(r,f) \le 2N(r,a;f) + S(r,f). \tag{3.17}$$

Therefore $N(r,a;f) \ne S(r,f)$. Also since $L^{(k+1)} \equiv f^{(1)}$. Then $L^{(k)} \equiv f + c$, where c is a constant. Also since $N(r,a;f) \ne S(r,f)$ and by hypothesis we get $c = 0$. Hence $L^{(k)} \equiv f$, which contradicts the initial supposition of Case 2.

SUBCASE 2.3. Let $L^{(k)} \equiv L^{(k+1)} \equiv f^{(1)}$. Then $L^{(k)} \equiv L^{(k+1)}$ implies $L^{(k)} = ce^z$. Hence $L^{(k)} \equiv L^{(k+1)} \equiv f^{(1)} = ce^z$ then $f = ce^z + d$, which implies f does not assume the values d and ∞, by Lemma 2.1 we get

$$\begin{aligned}T(r,f) &\le \overline{N}(r,0;f-a) + \overline{N}(r,0;f-\infty) + \overline{N}(r,0;f-d) + S(r,f)\\ &\le \overline{N}(r,a;f) + S(r,f).\end{aligned}$$

This implies $\overline{N}(r,a;f) \neq S(r,f)$. Also since $N_A(r,a;f) + N_B(r,a;L^{(k)}) = S(r,f)$ and $f = ce^z + d = L^{(k)} + d$ we see that $\overline{E}(r,a;f) \cap \overline{E}(r,a;L^{(k)}) \neq \emptyset$ this implies $d = 0$ and so $f \equiv L^{(k)}$, we arrive at a contradiction. This completes the proof of the theorem.

4. An open problem. Is it possible to replace the set B of hypothesis (i) of Theorem 1.1 by $B = \overline{E}(a;L^{(k)}) \setminus \{\overline{E}(a;f^{(1)}) \cap \overline{E}(a;L^{(k+1)})\}$?

Acknowledgement. The author is thankful to the referee for carefully reading the manuscript and giving suggestions towards improvement of the paper.

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